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Advanced Linear Algebra (MA 409)
Problem Sheet - 1

Vector Spaces

1. Let S be any nonempty set and F be any field, and let $\mathcal{F}(S,F)$ denote the set of all functions from S to F. Define vector addition and scalar multiplication on $\mathcal{F}(S,F)$ over F as follows :

   For $f,g \in \mathcal{F}(S,F)$ and $\alpha \in F$, $(f+g)(s) = f(s)+g(s)$ and $(cf)(s) = cf(s)$ for each $s \in S$. Prove that $\mathcal{F}(S,F)$ is a vector space over the field F with respect to the operations defined as above.

2. Let F be a field and $P(F)$ denote the set of polynomials with coefficients from the field F. With respect to the usual addition of polynomials and scalar multiplication, prove that $P(F)$ is a vector space over F.

3. Label the following statements are true or false.
   (a) In any vector space, $ax = bx$ implies that $a = b$.
   (b) In any vector space, $ax = ay$ implies that $x = y$.
   (c) In $P(F)$, only polynomials of the same degree may be added.
   (d) If f and g are polynomials of degree n, then $f+g$ is a polynomial of degree n.
   (e) Two functions in $\mathcal{F}(S,F)$ are equal if and only if they have the same value at each element of S.

4. Let V denote the set of ordered pairs of real numbers. If (a_1,a_2) and (b_1,b_2) are elements of V and $c \in \mathbb{R}$, define $(a_1,a_2)+(b_1,b_2) = (a_1+b_1, a_2+b_2)$ and $c(a_1,a_2) = (ca_1,a_2)$. Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

5. Let F be a field and let $M_{m\times n}(F)$ denote the set of all $m \times n$ matrices with entries are from the field F. Define vector addition and scalar multiplication on $M_{m\times n}(F)$ over F as follows :

   For $A,B \in M_{m\times n}(F)$ and $\alpha \in F$, $(A+B)_{ij} = A_{ij}+B_{ij}$ and $(\alpha A)_{ij} = \alpha A_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$. Prove that $M_{m\times n}(F)$ is a vector space over the field F with respect to the operations defined as above.

6. Let $V = \{(a_1,a_2) : a_1,a_2 \in F\}$, where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1,a_2) \in V$, define $c(a_1,a_2) = (a_1,0)$. Is V a vector space over F with these operations? Justify your answer.

7. Let $V = \{(a_1,a_2) : a_1,a_2 \in \mathbb{R}\}$. For $(a_1,a_2),(b_1,b_2) \in V$ and $c \in \mathbb{R}$. In each of the following, is V a vector space over $\mathbb{R}$ with these operations defined below? Justify your answer.
   (a) $(a_1,a_2)+(b_1,b_2) = (a_1+2b_1, a_2+3b_2)$ and $c(a_1,a_2) = (ca_1,ca_2)$.
   (b) $(a_1,a_2)+(b_1,b_2) = (a_1+b_1, a_2b_2)$ and $c(a_1,a_2) = (ca_1,ca_2)$.
   (c) $(a_1,a_2)+(b_1,b_2) = (a_1+b_1, 0)$ and $c(a_1,a_2) = (ca_1,0)$.
   (d) $(a_1,a_2)+(b_1,b_2) = (a_1+b_1, a_2+b_2)$ and $c(a_1,a_2) = (ca_1,a_2)$.

8. Let $V = \{(a_1,a_2) : a_1,a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinatewise, and for (a_1,a_2) in V and $c \in \mathbb{R}$, define

$$c(a_1,a_2) = \begin{cases} (0,0) & \text{if } c = 0. \\ (ca_1, a_2/c) & \text{if } c \neq 0. \end{cases}$$

   Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

9. How many matrices are there in the vector space $M_{m\times n}(\mathbb{Z}_2)$?

10. If V is a vector space over the field F, Verify that $(a+b)+(c+d) = [b+(c+a)]+d$ for all vectors a, b, c and d in V.

11. On $\mathbb{R}^n$, define two operations $x \oplus y = x - y$ and $c.x = -cx$. The operations on the right are the usual ones, which of the axioms for a vector space are satisfied by $(\mathbb{R}^n, \oplus, .)$?

12. Let V be the set of all complex-valued functions f on the real line such that (for all t in $\mathbb{R}$) $f(-t) = \overline{f(t)}$. The bar denotes complex conjugation. Show that V with the operations

$$\begin{aligned}(f+g)(t) &= f(t)+g(t)\\ (cf)(t) &= cf(t)\end{aligned}$$

is a vector space over the field of real numbers. Give an example of a function in V which is not real-valued.

13. Let W be of all ordered triplets (x_1, x_2, x_3) of real numbers such that $\frac{x_1}{3} = \frac{x_2}{4} = \frac{x_3}{2}$. Is W is a real vector space (vector space over $\mathbb{R}$) with respect to the usual operations in $\mathbb{R}^3$.

14. Is $\mathbb{R}$ with usual addition and multiplication a vector space over the filed of rational numbers?

15. Does the power set of a set Ω (all subsets of Ω) form a vector space over $F = \{0, 1\}$ with the operations given below? The sum of A and B is defined to be their symmetric difference :

$$A \Delta B = (A - B) \cup (B - A).$$

The scalar multiple αA is defined to be A if $\alpha = 1$ and $\emptyset$ (the null set) if $\alpha = 0$. Also find which of the axioms will be violated if addition of vectors is changed to $A + B = A \cup B$.

16. In each of the following, find precisely which axioms in the definition of a vector space are violated. Take $V = \mathbb{R}^2$ and $F = \mathbb{R}$ throughout.

    (a) $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$ and $c(a_1, a_2) = (ca_1, 0)$
    (b) $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, a_2) = (ca_1, 0)$
    (c) $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, a_2) = (ca_1, 2ca_2)$
    (d) $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and $c(a_1, a_2) = (c + a_1, c + a_2)$.

17. Show that the set of all positive real numbers forms a vector space over $\mathbb{R}$ if the sum of x and y is defined to be the usual product xy and α times x is defined to be x^α.

18. In the vector space F^3 where $F = \mathbb{Z}_3$, compute : $(1, 1, 2) + (0, 2, 2)$, the negative of $(0, 1, 2)$ and $2(1, 1, 2)$.

19. If G is a field and $F \subseteq G$ forms a subfield, show that G is a vector space over F. (What are the operations of this vector space?)

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